Jason Ferguson

Solutions to Homework Assignment #1
for MATH 54.002.

My solutions may not be the best way to solve these problems, and in fact you may find solutions that are better than mine. Congratulations if you do!

Section 1.1

1. $x_1 + 5x_2 = 7$
$-2x_1 - 7x_2 = -5.$

$$\begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix} \xrightarrow[\text{to Row 2}]{\text{Add } 2\times(\text{Row 1})} \begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix} \xrightarrow{\text{Multiply Row 2 by } \frac{1}{3}} \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow[\text{to Row 1}]{\text{Add } (-5)\times(\text{Row 2})} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix}$$

Answer is $\boxed{(x_1, x_2) = (-8, 3)}$.

3. Need to solve $x_1 + 5x_2 = 7$ and $x_1 - 2x_2 = -2$.

$$\begin{bmatrix} 1 & 5 & 7 \\ 1 & -2 & -2 \end{bmatrix} \xrightarrow[\text{to Row 2}]{\text{Add } (-1)\times(\text{Row 1})} \begin{bmatrix} 1 & 5 & 7 \\ 0 & -7 & -9 \end{bmatrix} \xrightarrow{\text{Multiply Row 2 by } -\frac{1}{7}} \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & \frac{9}{7} \end{bmatrix}$$

$$\xrightarrow[\text{to Row 1}]{\text{Add } (-5)\times(\text{Row 2})} \begin{bmatrix} 1 & 0 & \frac{4}{7} \\ 0 & 1 & \frac{9}{7} \end{bmatrix} \quad \left(7 - 5\cdot\frac{9}{7} = \frac{49}{7} - \frac{45}{7} = \frac{4}{7}\right)$$

Answer is $\boxed{(x_1, x_2) = (\frac{4}{7}, \frac{9}{7})}$.

5. $\boxed{\text{Add } (-5)\times \text{Row 3 to Row 1.}}$
$\boxed{\text{Add } 3\times \text{Row 3 to Row 2.}}$

7. $$\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{\text{Switch Rows 3 and 4}} \begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Add } 2\times(\text{Row 4}) \text{ to Row 3}]{\substack{\text{Add } 4\times(\text{Row 4}) \text{ to Row 1} \\ \text{Add } (-3)\times(\text{Row 4}) \text{ to Row 2}}} \begin{bmatrix} 1 & 7 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\xrightarrow[\text{Add Row3 to Row2}]{\text{Add }(-3)\times(\text{Row3})\text{ to Row1}} \begin{bmatrix}1&7&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \xrightarrow{\text{Add }(-7)\times\text{Row2 to Row1}} \boxed{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}$

The system has $\boxed{\text{no solutions.}}$ (You don't need any row reductions to figure this out, since the original matrix has [0 0 0 1] as a row. But the problem said to row-reduce.)

9. $\begin{bmatrix}1&-1&0&0&-4\\0&1&-3&0&-7\\0&0&1&-3&-1\\0&0&0&2&4\end{bmatrix} \xrightarrow{\text{Multiply Row 4 by }\frac{1}{2}} \begin{bmatrix}1&-1&0&0&-4\\0&1&-3&0&-7\\0&0&1&-3&-1\\0&0&0&1&2\end{bmatrix}$

$\xrightarrow{\text{Add }3\times(\text{Row 4})\text{ to Row 3}} \begin{bmatrix}1&-1&0&0&-4\\0&1&-3&0&-7\\0&0&1&0&5\\0&0&0&1&2\end{bmatrix} \xrightarrow{\text{Add }3\times(\text{Row 3})\text{ to Row 2}} \begin{bmatrix}1&-1&0&0&-4\\0&1&0&0&8\\0&0&1&0&5\\0&0&0&1&2\end{bmatrix}$

$\xrightarrow{\text{Add Row2 to Row1}} \boxed{\begin{bmatrix}1&0&0&0&4\\0&1&0&0&8\\0&0&1&0&5\\0&0&0&1&2\end{bmatrix}}$

Solution is $\boxed{(x_1, x_2, x_3, x_4) = (4, 8, 5, 2)}$.

11. $\begin{bmatrix}0&1&4&-5\\1&3&5&-2\\3&7&7&6\end{bmatrix} \xrightarrow{\text{Switch Rows 1 and 2}} \begin{bmatrix}1&3&5&-2\\0&1&4&-5\\3&7&7&6\end{bmatrix}$

$\xrightarrow{\text{Add }(-3)\times\text{ Row 1 to Row 3}} \begin{bmatrix}1&3&5&-2\\0&1&4&-5\\0&-2&-8&12\end{bmatrix} \xrightarrow{\text{Add }2\times\text{Row2 to Row3}} \begin{bmatrix}1&3&5&-2\\0&1&4&-5\\0&0&0&2\end{bmatrix}$.

$\boxed{\text{No solutions}}$

13. $\begin{bmatrix}1&0&-3&8\\2&2&9&7\\0&1&5&-2\end{bmatrix} \xrightarrow{\text{Add }(-2)\times\text{Row1 to Row 2}} \begin{bmatrix}1&0&-3&8\\0&2&15&-9\\0&1&5&-2\end{bmatrix} \xrightarrow{\text{Switch Rows 2 and 3}} \begin{bmatrix}1&0&-3&8\\0&1&5&-2\\0&2&15&-9\end{bmatrix}$

$\xrightarrow{\text{Add }(-2)\times\text{Row2 to Row3}} \begin{bmatrix}1&0&-3&8\\0&1&5&-2\\0&0&5&-5\end{bmatrix} \xrightarrow{\text{Multiply Row3 by }\frac{1}{5}} \begin{bmatrix}1&0&-3&8\\0&1&5&-2\\0&0&1&-1\end{bmatrix}$

$\xrightarrow[\text{Add }(-5)\times\text{Row3 to Row2}]{\text{Add }3\times\text{Row3 to Row1}} \begin{bmatrix}1&0&0&5\\0&1&0&3\\0&0&1&-1\end{bmatrix}$

Solution is $\boxed{(x_1, x_2, x_3) = (5, 3, -1)}$

15. $\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix} \xrightarrow[\text{to Row 4}]{\text{Add } (-3)\times(\text{Row }1)} \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$

$\xrightarrow[\text{to Row 3}]{\text{Add } 2\times(\text{Row }2)} \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix} \xrightarrow[\text{by } \frac{1}{3}]{\text{Multiply Row 3}} \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -\frac{4}{3} & \frac{7}{3} \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$

$\xrightarrow[\text{to Row 4}]{\text{Add } 9\times(\text{Row }3)} \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -\frac{4}{3} & \frac{7}{3} \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix}$ $\quad \left(\begin{array}{l} 9(-\frac{4}{3}) + 7 = -12 + 7 = -5 \\ 9(\frac{7}{3}) + (-11) = 21 + (-11) = 10 \end{array} \right)$

$\xrightarrow[\text{by } -\frac{1}{5}]{\text{Multiply Row 4}} \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -\frac{4}{3} & \frac{7}{3} \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$

System is $\boxed{\text{consistent}}$.

20. $\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix} \xrightarrow[\text{to Row 2}]{\text{Add } 2\times(\text{Row }1)} \begin{bmatrix} 1 & h & -3 \\ 0 & 2h+4 & 0 \end{bmatrix}$

This is $\boxed{\text{consistent for all } h}$, since $(x_1, x_2) = (-3, 0)$ is always a solution.

Jason Ferguson

28. $\begin{bmatrix} a & b & f \\ c & d & g \end{bmatrix} \xrightarrow[\text{by } \frac{1}{a}]{\text{Multiply Row 1}} \begin{bmatrix} 1 & \frac{b}{a} & \frac{f}{a} \\ c & d & g \end{bmatrix}$

$$\xrightarrow[\text{to Row 2}]{\text{Add } (-c)\times(\text{Row 1})} \begin{bmatrix} 1 & \frac{b}{a} & \frac{f}{a} \\ 0 & d-\frac{bc}{a} & g-\frac{cf}{a} \end{bmatrix}.$$

If $d-\frac{bc}{a}=0$, then picking $f=0$ and $g=1$ would make Row 2 equal to $[0 \quad 0 \quad 1]$, which means the system is inconsistent.

If $d-\frac{bc}{a}\neq 0$, then we can divide by it:

$$\xrightarrow[\text{by } \frac{1}{d-\frac{bc}{a}} = \frac{a}{ad-bc}]{\text{Multiply Row 2}} \begin{bmatrix} 1 & \frac{b}{a} & \frac{f}{a} \\ 0 & 1 & \frac{ag-cf}{ad-bc} \end{bmatrix},$$

This last system is consistent for all f and g.

$\boxed{\text{So we know that } d-\frac{bc}{a}\neq 0\text{, and that is all we know}}$

(There are other ways of writing $d-\frac{bc}{a}\neq 0$. For example, since $a\neq 0$, you can multiply by a to get that $d-\frac{bc}{a}\neq 0$ means the same thing as $ad-bc\neq 0$.
In fact, $ad-bc\neq 0$ is still the answer even when $a=0$, but you need different steps in your algebra to figure this out.)

Section 1.2

1. $\boxed{\text{Only a and b are in reduced echelon form.}}$
$\boxed{\text{Only a, b, and d are in echelon form.}}$

Jason Ferguson

3. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \xrightarrow[\text{Add } (-6)\times(\text{Row }1) \text{ to Row }3]{\text{Add } (-4)\times(\text{Row }1) \text{ to Row }2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix}$

$\xrightarrow{\text{Add}(-\frac{5}{3})\times \text{Row }2 \text{ to Row }3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{\text{Multiply Row}2 \text{ by } (-\frac{1}{3})} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{\text{Add } (-2)\times \text{Row }2 \text{ to Row}3} \boxed{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$

Pivot positions are: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ in original matrix and $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ in the reduced matrix. The pivot colums are columns $\boxed{\text{one and two.}}$

5. The answer is $\boxed{\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}.}$ Since the matrix is nonzero, its first row is nonzero. If the first entry in the first row is nonzero, then the entry below it is zero, but the entries in the second column can be anything. If the first entry in the first row is zero, then the second entry in the first row is nonzero and every other entry is zero.

7. $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \xrightarrow[\text{to Row }2]{\text{Add } (-3)\times\text{Row }1} \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}$

$\xrightarrow[\text{by } -\frac{1}{5}]{\text{Multiply Row}3} \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[\text{to Row }1]{\text{Add } (-4)\times\text{Row}2} \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$

Solution is: $\boxed{\begin{array}{l} x_1 = -3x_2 - 5 \\ x_2 \text{ is free} \\ x_3 = 3 \end{array}}$

Jason Ferguson

9. $\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -6 \end{bmatrix} \xrightarrow{\text{Switch Rows 1 and 2}} \begin{bmatrix} 1 & -2 & 7 & -6 \\ 0 & 1 & -6 & 5 \end{bmatrix}$

$\xrightarrow{\text{Add } 2\times(\text{Row }2) \text{ to Row } 1} \begin{bmatrix} 1 & 0 & -5 & 4 \\ 0 & 1 & -6 & 5 \end{bmatrix}$

Solution is $\boxed{\begin{array}{l} x_1 = 5x_3 + 4 \\ x_2 = 6x_3 + 5 \\ x_3 \text{ is free} \end{array}}$

11. $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} \xrightarrow[\text{Add } 2\times(\text{Row } 1) \text{ to Row } 3]{\text{Add } 3\times(\text{Row } 1) \text{ to Row } 2} \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{\text{Multiply Row 1 by } \frac{1}{3}} \begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

Solution is $\boxed{x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3, \quad x_2 \text{ and } x_3 \text{ are free}}$.

13. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Add Row3 to Row1}} \begin{bmatrix} 1 & -3 & 0 & 0 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{\text{Add } 3\times(\text{Row } 2) \text{ to Row } 1} \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 5 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Solution is $\boxed{x_1 = 3x_5 + 5, \ x_2 = 4x_5 + 1, \ x_3 \text{ is free}, \ x_4 = -9x_5 + 4, \ x_5 \text{ is free}}$

15(a) The system is $\boxed{\text{consistent, with a unique solution}}$. It is consistent because the squares are pivots so the rightmost column is not a pivot column. It has a unique solution because putting the matrix into reduced row echelon form will turn the pivots into 1's, and all entries directly above and below the pivots into zeroes in other words, into

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$$

which has the unique solution $x_1 = a$, $x_2 = b$, $x_3 = c$.

(b) The system is $\boxed{\text{inconsistent}}$ as it has a row of the form $[0 \ 0 \ 0 \ 0 \ b]$ where b is nonzero.

23. Since a matrix can have at most one pivot position in each row, a matrix can't have more pivot positions than rows, so it also can't have more pivot columns than it has rows.

The augmented matrix of this system is 3×6, so it can have at most three pivot columns. The three pivot columns of the coefficient matrix must also be pivot columns of the augmented matrix (since the row operations that reduce the coefficient matrix also reduces the augmented matrix). Therefore, some three of the leftmost 5 columns of the augmented matrix are pivot columns. That means the rightmost column of the augmented matrix can't be a pivot column, so the system is $\boxed{\text{consistent}}$

24. The rightmost column is a pivot column is a pivot column, so the system is $\boxed{\text{inconsistent.}}$

25. (The idea is similar to that of Ex. 1.2.23.)

Let m and n be the number of rows and columns of the coefficient matrix. Then since every row has a pivot position, and no row can have more than one pivot position, the coefficient matrix has m pivot positions. No column can have more than one pivot position, so the coefficient matrix has m pivot columns.

These m pivot columns must also be pivot columns of the augmented matrix, so some m of the leftmost n columns of the augmented matrix are pivot columns. The augmented matrix is $m\times(n+1)$, and no matrix can have more pivot columns than rows, so the rightmost column of the augmented matrix is not a pivot column. Therefore, the system is $\boxed{\text{consistent}}$

26. Since the coefficient matrix has three pivots, the augmented matrix has

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}$$

as its echelon form. Then the reduced echelon form must be

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix},$$

which has $x_1=a$, $x_2=b$, and $x_3=c$ as its only solution.

Jason Ferguson

30. $\boxed{\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}}$

The system is inconsistent because it has no solutions.
(Nothing that satisfies the first equation can also satisfy the second.)
The system is underdetermined because it has more unknowns than equations. (There are many other examples.)
(The word "underdetermined" is defined in Exercise 29 of section 1.2; the exercise before this one.)